AN (INCOMPLETE) INTRODUCTION TO GÖDEL'S INCOMPLETENESS THEOREM

Corentin Dupont

University of Trento, 19/09/2013

Table of Contents

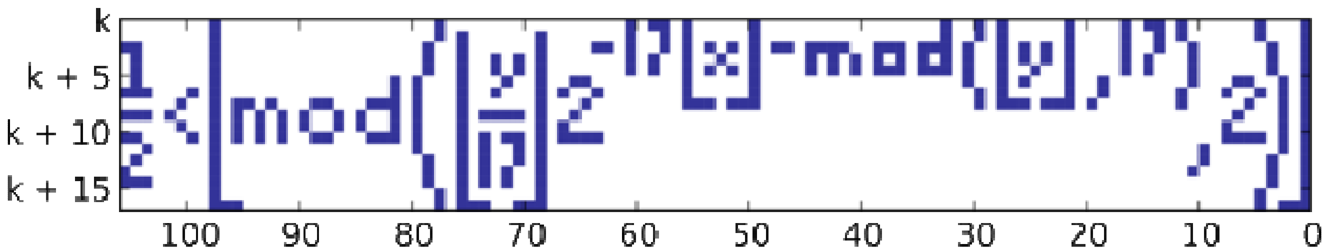
SELF REFERENCE

- Any object that refers to itself or its own referent
  - Quines in computing
  - Recursion in programming
  - Self referential statements in linguistics

Tupper's Self Referential Formula

$$\frac{1}{2} < \left\lfloor \mathrm{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \mathrm{mod}(\lfloor y \rfloor, 17)}, 2 \right) \right\rfloor$$

Graphical plot of the above formulae:



Self Reference

- Liar's Paradox
  - "This sentence is false.“
  - “The next sentence is false. The previous sentence is true."
- Epimenides Paradox
  - “All Cretans are liars”
- Quines:

```
#include<stdio.h>
main(){char*a="#include<stdio.h>%cmain(){char*a=%c%s%c;printf(
a,10,34,a,34);}";printf(a,10,34,a,34);}
```

"yields falsehood when preceded by its quotation"
yields falsehood when preceded by its quotation.

RUSSEL'S PARADOX

- In certain village, there is one barber who shaves every man except those men who shave themselves.

- Who shaves the barber?

Russel's paradox

- S = the set of all sets not containing themselves
- Is S containing itself?
- If S is an element of S, then S is a member of itself and should not be in S.
- If S is not an element of S, then S is not a member of itself, and should be in S.

PRINCIPIA MATHEMATICA

- Whitehead and Russell (1910– 1913)
  - Three Volumes, 2000 pages
- Attempted to axiomatize mathematical reasoning
  - Define mathematical entities (like numbers) using logic
  - Derive mathematical “truths” by following mechanical rules of inference
  - Formal system should be consistent and complete

DEFINITIONS

- For a formal system:
- It is consistent if it never proves a statement to be both true and false.
- It is complete if every "true" statement can be proved.

Ban Self-Reference?

- *Principia Mathematica* attempted to resolve this problem by banning self-reference
- Every set has a type
  - The lowest type of set can contain only "objects", not "sets"
  - The next type of set can contain objects and sets of objects, but not sets of sets

Kurt Gödel

Kurt Gödel, born April 28, 1906, was an Austrian - American logician , mathematician and philosopher . One of the most significant logicians of all time, Gödel made an immense impact upon scientific and philosophical thinking in the 20th century.

Gödel is best known for his two incompleteness theorems, published in 1931 when he was 25 years of age, one year after finishing his doctorate at the University of Vienna.

Gödel's Theorem

In any interesting formal system, there are statements that cannot be proved either true or false.

Interesting = consistent and capable of expressing elementary integer arithmetic

DOUGLAS HOFSTADTER'S PROOF

- Defines a formal axiomatic system called Typographical Number Theory (TNT)
- Assumes TNT to be complete
- Leads to contradiction

Typographical Number Theory (TNT)

- Mathematical Symbols : +, *, =
- Variables : a, a', a'', a''', ...
- Logical Symbols : A, E, v,^, ~
- Numbers : 0, S0, SS0, SSS0, ...

- Statements
  - S0 + SS0 = SSS0

- Axioms
  - Aa:~Sa=0
  - Aa:(a+0)=a
  - Aa:Aa':(a+Sa')=S(a+a')
  - Aa:(a*0)=0
  - Aa:Aa':(a*Sa')=((a*a')+a)

TNT

- Rules
  - ~~ can be removed from any statement
  - Au:~ and ~Eu: are interchangeable anywhere inside a string.
  - …

- TNT-Theorem: TNT-Statement which can be inferred from TNT-Axioms and TNT-Rules

- Example:
  - S0+S0=SS0
  - Aa:Aa':(a+a')=(a'+a) (Commutativity)
  - ~Ea:a*a=SS0 (No root of 2)
  - ~(Ea':Ea'':SSa'*SSa'' = SSSSS0) (Primality of 5)

- Assumption: Any true statement can be derived in TNT

GÖDEL'S STATEMENT IN TNT

- G: “This statement is not a theorem of TNT”
- If G is false, then it is a theorem of TNT. Then we have a valid theorem which is false, this is not possible.
- Hence G is true and it is not a theorem of TNT
- Hence there is a true statement in TNT which is not provable in TNT.

GÖDEL NUMBERING

- Replace every symbol with three digit unique number
- 0 - 666
- a - 262
- S - 123
- = - 111
- and so on ...

- Every TNT-Statement thus has a unique Gödel number
- Rules become functions on natural numbers

[Kurt Gödel, "On formally undecidable propositions of Principia Mathematica and related systems", Monatshefte für Mathematik und Physik, 1931]

GÖDEL NUMBERING

- Every TNT-Statement thus has a unique Gödel number
  - Example

    ~ E a : a * a = SS0
    223 333 262 636 262 236 262 111 123 123 666

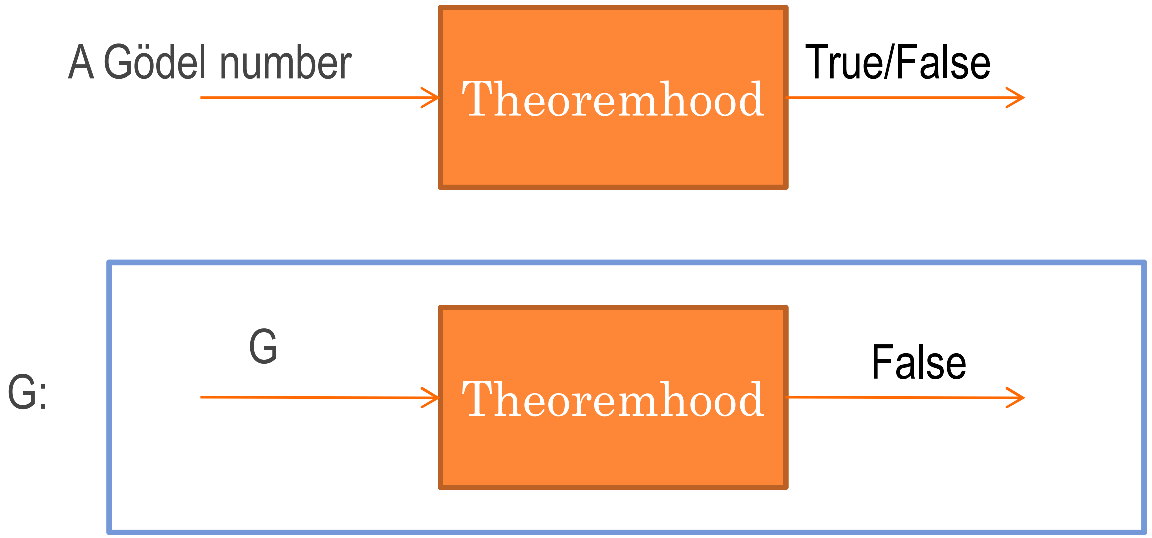
- Rules become functions on natural numbers
  - Example

    Dummy Rule: Whenever a string ends in the symbol "000", you can replace that symbol with "005".

    The same fake rule, written differently: Whenever a number is a multiple of 1000, you can add 5 to it.

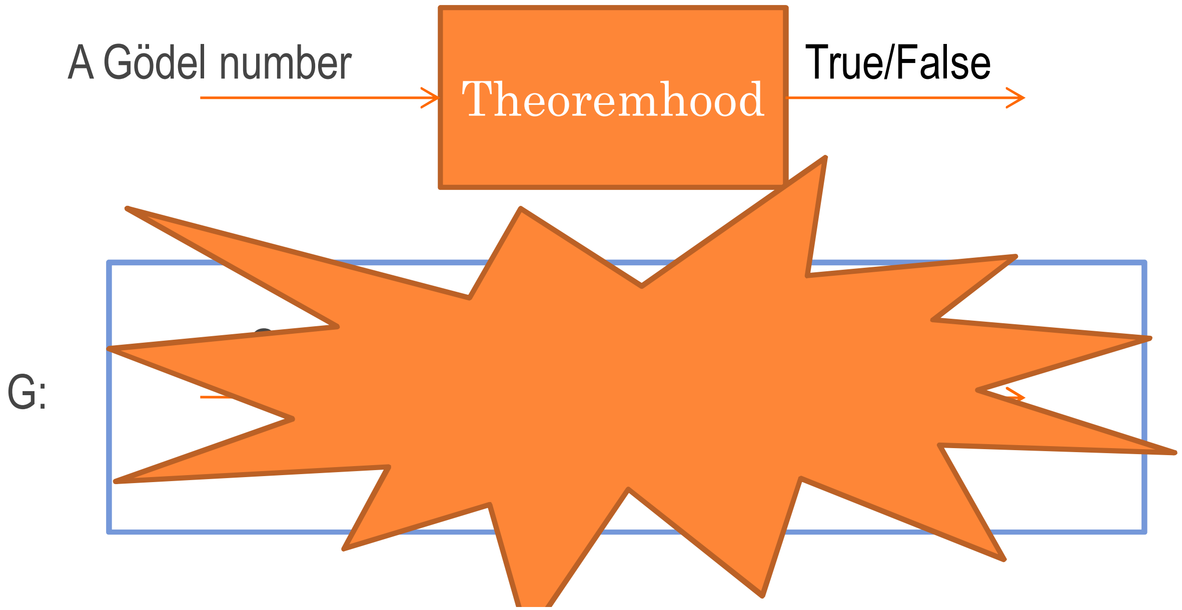
THEOREMHOOD

- Theoremhood of a number: Being derivable by applying certain functions (Rules) on certain initial numbers (Axioms)
- Benefit of Gödel Numbering: A statement can talk about a statement of TNT by using its Gödel number
- Difficulty: Gödel number of a statement is longer than the statement itself



THEOREMHOOD

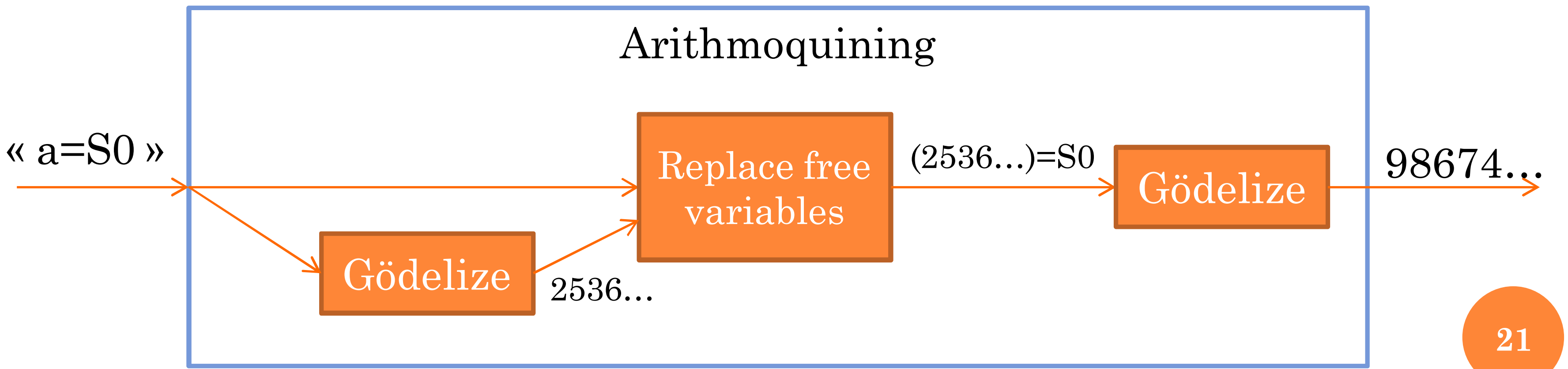
- Theoremhood of a number: Being derivable by applying certain functions (Rules) on certain initial numbers (Axioms)
- Benefit of Gödel Numbering: A statement can talk about a statement of TNT by using its Gödel number
- Difficulty: Gödel number of a statement is longer than the statement itself



ARITHMOQUINING

- Mechanism for writing a TNT sentence about another TNT sentence
- Take a sentence, replace all free variables with the Gödel number of the sentence.

- Example:
- T: a=S0
  A: “Sentence T” =S0
  where “Sentence T” is the Gödel number of T



FINALLY G IS HERE!!!

- G: “This statement is not a theorem of TNT”
- T: “The arithmoquine of a is not a valid TNT theorem-number”
- G: “The arithmoquine of Sentence T is not a valid TNT theorem-number”
- G uses "arithmoquine of Sentence T" to intelligently refer to itself
- G is true and not a theorem of TNT as shown before

The Second Incompleteness Theorem

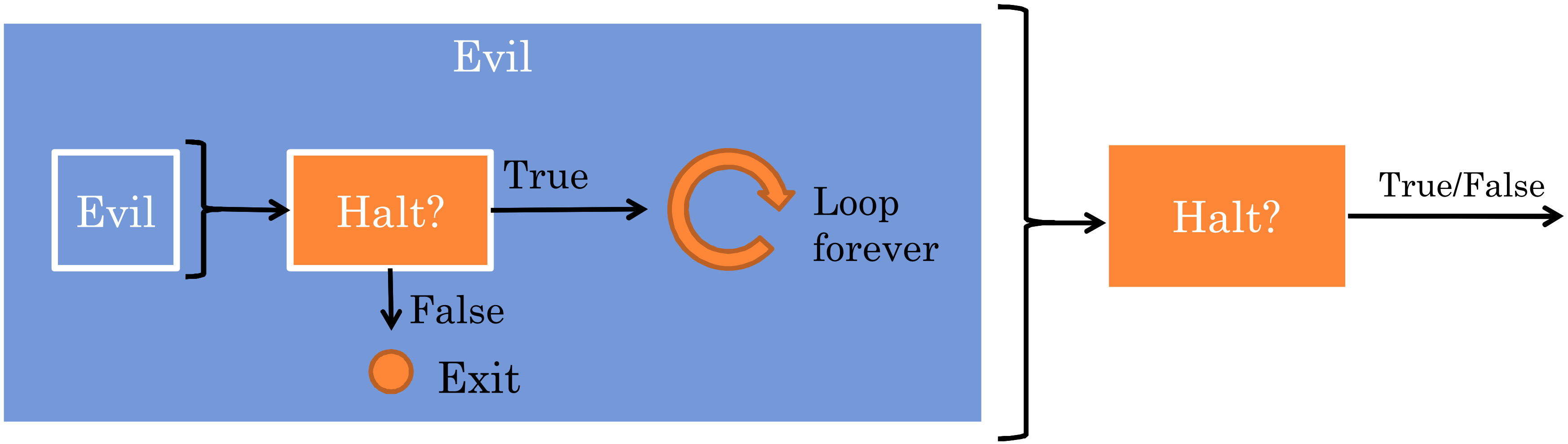
- Every formal system G strong enough to encode arithmetic in N is either inconsistent or cannot prove its own consistency.

Relationship with Computability

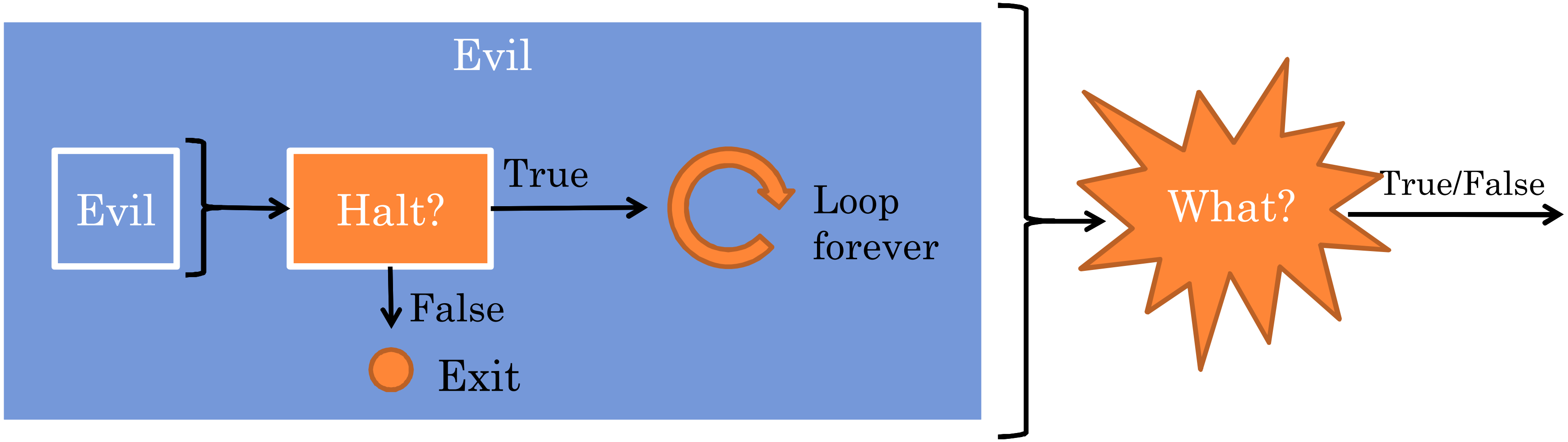
- the *halting problem* is undecidable

The programming language must be general enough to be equivalent to a Turing machine

THE HALTING PROBLEM



THE HALTING PROBLEM



IMPLICATIONS

- The old mathematical program of putting math on secure logical foundations has its limits: you never can be sure it won't be undermined.
- Some philosophers (e.g., John Lucas) argue that Gödel's Theorem proves minds are not the same as machines, and that computers will never achieve an artificial intelligence equal to the human mind.
- There has been a wealth of new true-but-unprovable statements discovered:
  - recognizing whether certain groups are isomorphic
  - recognizing whether certain manifolds are homeomorphic
  - proving that a computer file is incompressible

SOME MISUNDERSTANDINGS…

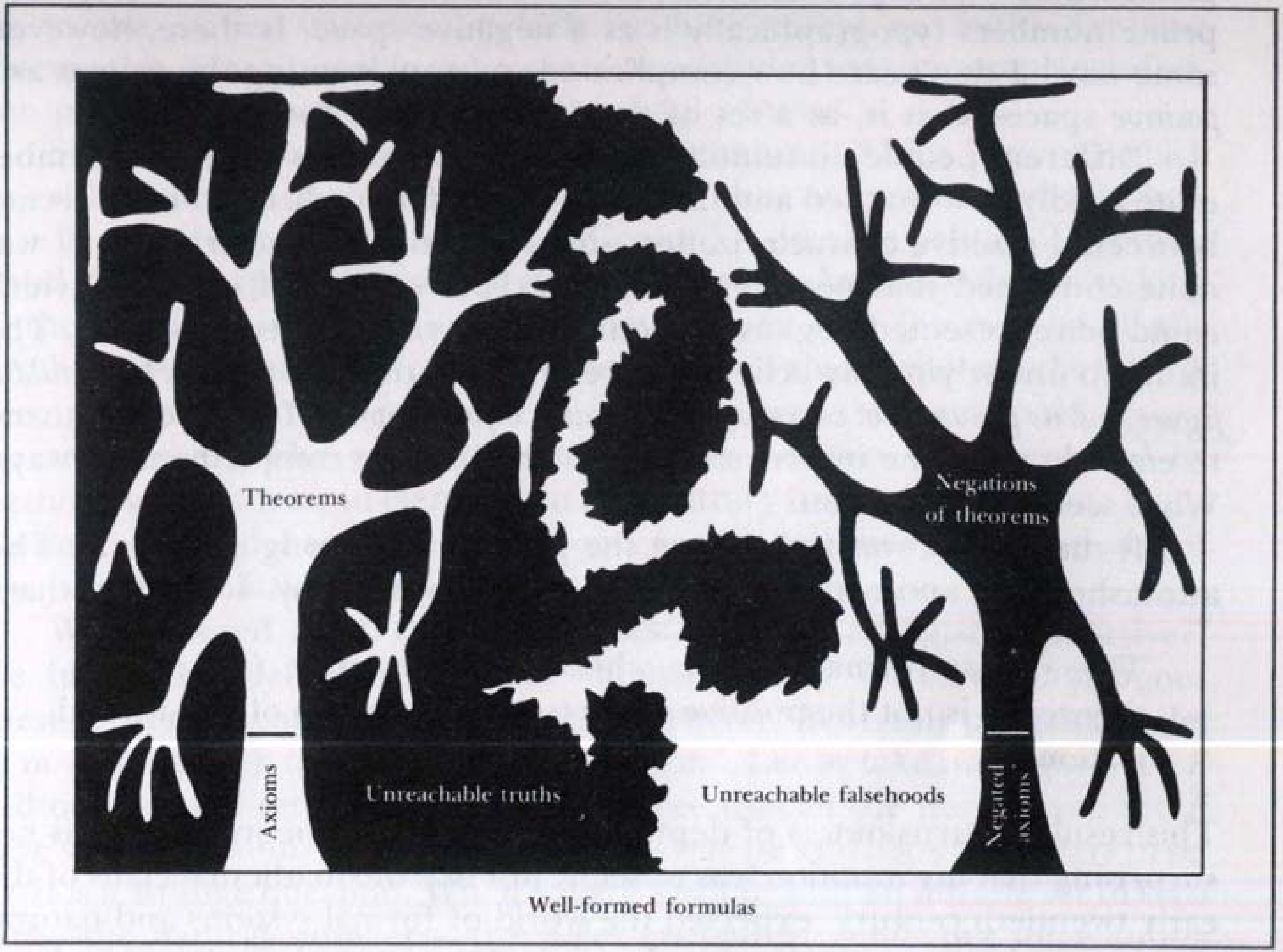
- "There are truths that cannot be proved"

Incorrect: "provability" is always relative to a formal system.

- "Every consistent formal system is incomplete"

Incorrect: some are consistent and complete, but…

- General "truth" is not a mathematical concept. A proof in logic is a syntactic derivation from the axioms
- The belief of the consistency of the axioms of ordinary mathematics comes from consensus, common sense, experience, tradition, etc., not from logic


Theorems
Negations of theorems
Axioms
Unreachable truths
Unreachable falsehoods
Negated axioms
Well-formed formulas

References

- Book: "Le Théorème de Gödel", Ernest Nagel
- Book: "Gödel, Escher, Bach: an Eternal Golden Braid", Douglas R. Hofstadter
- Paper: J.R. Lucas, "Minds, Machines and Gödel", in Minds and Machines, ed. Alan R. Anderson http://users.ox.ac.uk/~jrlucas/Godel/mmg.html

- Pictures: Gödel, Escher, Bach, Douglas R. Hofstadter
- Wikipedia

Back-up

- It is often said that Gödel demonstrated that there a truths that cannot be proved. This is incorrect, “provability” is always relative to a formal system
- If an arithmetic sentence was undecidable in ZFC we still could select a stronger set of axioms valid as foundations and try to prove it there
- That is not to say that such a switch would bewithout controversy

- The incompleteness theorem does not imply that every consistent formal system is incomplete
- The theory of the real numbers is complete, and it comprises the arithmetic of real numbers
- Although the natural numbers are a subset of the real numbers they are not definable within the theory of real numbers, and so the premise of the incompleteness theorem do not hold
- “Truth” is not a mathematical concept, and so it is normally avoided by mathematicians, a proof in logic is a syntactic derivation and has nothing to do with the “truth” or “falsehood” of a sentence